JHEP HYPER VERSION

Received: *January 27, 1999*, Accepted: *March 23, 1999*

Note on the quantum mechanics of M-Theory

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Abstract: We observe that the existence of black holes limits the extent to which M-Theory (or indeed any quantum theory of gravity) can be described by conventional quantum mechanics. Although there is no contradiction with the fundamental properties of quantum mechanics, one can prove that expectation values of Heisenberg operators at fixed times cannot exist in an ordinary asymptotic Lorentz frame. Only operators whose matrix elements between the vacuum and energy eigenstates with energy greater than the Planck scale are artificially cut off, can have conventional Green's functions. This implies a Planck scale cutoff on the possible localization of measurements in time. A similar behavior arises also in "little string theories". We argue that conventional quantum mechanics in light cone time is compatible with the properties of black holes if there are more than four non-compact flat dimensions, and also with the properties of "little string theories". We contrast these observations with what is known about M-Theory in asymptotically Anti-de Sitter spacetimes.

Keywords: M-Theory, Black Holes.

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Contents

1. Introduction

It is universally believed that M-Theory is described by ordinary quantum mechanics. In this paper we will present evidence to the contrary. However, the modification of quantum mechanics we propose is very mild and indeed the formalism we will use to investigate this question is the standard formalism of the quantum theory. The only conventional axiom that we have to drop is the implicit one which allows us to define Heisenberg operators, given an initial Hilbert space and hamiltonian. We argue that in asymptotically Minkowski space time, this cannot be done in M-Theory (or any other candidate quantum theory of gravity) for any timelike asymptotic Killing vector. This is a consequence of the Bekenstein-Hawking formula for black hole entropy. The same considerations tell us why it *is* possible to construct a standard hamiltonian quantum mechanics for M Theory in asymptotically anti-de Sitter (AdS) spaces.

We argue that when the number of Minkowski dimensions is greater than four, hamiltonian quantum M-Theory in the light cone frame is still sensible. We also note in passing that the success of light cone string theory (in contrast with temporal gauge quantization of the string) is related to the Hagedorn density of states of the string.

In precisely four noncompact dimensions, the light cone formalism fails to cope with the density of states of black holes. This suggests that there may not even be a light cone hamiltonian formalism for nonperturbative M-Theory in this framework.

Matrix Theory has already provided evidence for the possible breakdown of the light cone description of M-Theory with four noncompact dimensions. However, in Matrix Theory this appeared to be related to the extra compactification of a lightlike circle. It is not clear to what extent these problems are related to the (much milder) difficulties we will point out here. A light-cone description of the four dimensional theory in terms of a non-standard quantum theory (such as a little string theory) is not ruled out by our considerations; in the Matrix Theory context such a description arises already when there are six non-compact dimensions.

We also use similar methods to analyze non-gravitational little string theories, and we conclude that they also do not have an ordinary quantum mechanical description in the usual time frame. However, they can (and do) have such a description in a light-cone frame.

2. The spectrum of a hamiltonian, and Heisenberg operators

In a generally covariant theory, the definition of time and time translation must be based on a physical object. In noncompact spacetime with appropriate asymptotic boundary conditions, appropriate physical objects are the frozen classical values of the asymptotic spacetime metric, and other fields. We will restrict attention to such asymptotically symmetric spaces in this paper.

The quantum theory lives in a Hilbert space which carries a representation of the asymptotic symmetry group, and among its generators we find the time translation operator.[1] In the Minkowski case there is, up to conjugation, a unique choice of time translation generator, while in the AdS case there are two interesting inequivalent choices.

Having chosen a hamiltonian, we are now ready to discuss Heisenberg operators, naively defined by

$$O(t) \equiv e^{iHt} O e^{-iHt} . \tag{2.1}$$

To test the meaning of this definition we compute the two point (Wightman) function; the ground state expectation value of the product of two operators at different times. By the usual spectral reasoning it has the form

$$W(t) = \langle 0|O^\dagger(0)O(t)|0\rangle = \int_0^\infty dE e^{-iEt} \rho_O(E) , \tag{2.2}$$

where the spectral weight is defined by:

$$\rho_O(E) = \sum_n \delta(E - E_n) |\langle 0|O|n\rangle|^2 \tag{2.3}$$

and is manifestly a sum of positive terms.

[1] For asymptotically Minkowski spaces we will ignore the full Bondi-Metzner-Sachs symmetry group in this paper and imagine that a particular Poincare subgroup of it has been chosen by someone wiser than ourselves. Perhaps one should think instead of the *Spi* group of asymptotic diffeomorphisms of spacelike infinity, where a natural Poincare subgroup exists.

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The crucial issue is now the convergence of this formal Fourier transform. In interacting quantum field theory, an operator localized in a volume V will typically have matrix elements between the vacuum and almost any state localized in V. The only restrictions will come from some finite set of global quantum numbers. At very high energies the density of states in volume V is controlled by the UV fixed point theory of which the full theory is a relevant perturbation. Scale invariance and extensivity dictate that it has the form

$$\rho(E) \sim e^{cV^{\frac{1}{d}}E^{\frac{d-1}{d}}} , \tag{2.4}$$

for some constant c, where d is the spacetime dimension. As a consequence, even if an operator O has matrix elements between the vacuum and states of arbitrarily high energy (and indeed, even if these matrix elements grow as the exponential of a power of the energy less than one, which is not typical), the Fourier integral converges and defines a distribution.[2]

In local field theory there are special operators whose matrix elements between the vacuum and high energy states are highly suppressed. These are operators which are linear functionals of local fields of fixed dimension at the UV fixed point. The spectral function of such fields has power law dependence on the energy (as opposed to the exponential dependence implied by (2.4)), which means that most of their matrix elements to most high energy states vanish rapidly with the energy. A typical operator localized in volume V is a nonlinear functional of such fields. Using the operator product expansion we can write a formal expansion of it in terms of the linear fields, but it will generically contain large contributions from fields of arbitrarily large dimension, and its spectral function will have the asymptotics of the full density of states.

Physically, we can understand the special role of local operators by first thinking of a massive field theory at relatively low energy. The intermediate states of energy E created by a local field will be outgoing scattering states of a number of massive particles bounded by E/m, whose momenta point back to the place where the field acts. When E is large these will be very special states among all those available in the same volume and with the same energy. A localized burst of energy will dissipate rather than thermalize. As E gets very large, this description becomes inadequate. However, note that if we choose a basis of local fields of fixed dimension in the UV conformal field theory, then any given field in the basis creates only states in a single irreducible representation of the conformal algebra. The degeneracy of these representations is much smaller than that of the full theory.

Now we want to ask the same question for M-Theory compactified to d asymptotically Minkowski spacetime dimensions. As in our discussion of field theory, we work in a fixed asymptotic Lorentz frame and discuss the time evolution operator

[2]This may be seen by analytic continuation to Euclidean space.

appropriate to that frame. We claim that the high energy density of states in superselection sectors with finite values of all Lorentz scalar charges, is dominated by d dimensional Schwarzschild black holes. The density of such black hole states is, by the Bekenstein-Hawking formula,

$$\rho_{BH} \sim e^{c(E/M_P)^{\frac{(d-2)}{(d-3)}}}, \tag{2.5}$$

where M_P is the d dimensional Planck mass and c is a constant. The Fourier integral no longer converges unless the matrix elements of the operator vanish rapidly for large E.[3] Note furthermore that if we employ an operator with an energy cutoff much larger than the Planck mass (the point above which the behavior (2.5) presumably sets in), then for all times longer than the inverse cutoff, the integral will be completely dominated by the states at the cutoff energy scale. Thus the only kinds of operators whose Green functions exhibit the usual property that the variation over a time scale ΔT probes excitations of energy $1/\Delta T$ are those with an energy cutoff below M_P.

The necessity of cutting off operators implies a non-locality in the physics of M-Theory at the time scale M_P^{-1}. Indeed, a probe of the system localized at time $t = t_0$ differs from one localized at $t = 0$ because it couples to the operator $\int dE e^{iEt_0} O(E)$ rather than the same integral with t_0 set to zero. If $t_0 \ll M_P^{-1}$ and $O(E)$ is cut off at $E \sim M_P$ then these operators are indistinguishable.

An important feature of the density of states (2.5), which distinguishes it from that of field theory, is its independence of the volume. This is related to the familiar instability of the thermal ensemble in quantum gravity to the formation of black holes [1]. In ordinary field theory, the typical state of very high energy E on a torus is a member of the translation-invariant thermal ensemble. However, as a consequence of the Jeans instability, any attempt to create a translationally invariant state with finite energy density in a theory containing gravity will fail. Once the size of a patch of finite energy density exceeds its Schwarzschild radius, it collapses into a black hole. So, the generic high energy state in M-Theory with four or more asymptotically flat dimensions is a single black hole.

This gives us some insight into the question raised in the previous footnote of why there are no analogs in M-Theory of local operators of fixed dimension which couple only to a few of the high energy states. In quantum gravity an operator carrying a very large energy E cannot create a state more localized than the corresponding Schwarzschild radius. The most localized states of a given energy are also the generic states of that energy and they are black holes.

[3] or unless the operator connects the vacuum only to a very small fraction of the black hole states. Given the thermal nature of the black hole ensemble, i.e. the fact that it consists primarily of states with the same gross properties, operators of the latter type do not correspond to very realistic probes of the system.

Some readers familiar with the AdS/CFT correspondence [2] will at this point be objecting that these considerations seem to contradict the successful description of M-Theory in AdS spaces by quantum field theories. Other AdS/CFT *cognoscenti* will have already recognized that in fact that correspondence is another example of the rules we have just promulgated. AdS spaces have two inequivalent types of interesting time evolutions, the Poincare and the global time. The appropriate black objects which dominate the high energy density of states for these two definitions of energy are near extremal black branes, and AdS Schwarzschild black holes respectively. Both of these have positive specific heat, that is the density of states is an exponential of a power of the energy less than one. This is completely consistent with quantum field theory, and of course the matching of the thermodynamics of these objects with that of conformal field theories [3] is one of the primary clues which led to the AdS/CFT correspondence.

Note that the major discrepancy between the behavior of the density of states in AdS and Minkowski spacetimes suggests that the extraction of flat space physics from that of AdS may be quite subtle. In [4] it was suggested that in appropriate regions of parameter space, the Hilbert space of the quantum field theories describing AdS physics contain an energy regime describing flat space black holes whose Schwarzschild radius is much smaller than the radius of curvature of AdS. Somehow one must find observables in the AdS theory which probe only this energy regime and reduce to the corresponding flat space S-matrix.

To summarize: we have argued that in asymptotically Minkowski space time, M-Theory cannot be described as a conventional quantum theory. Although the violation of the rules of quantum mechanics appears mild, we would like to emphasize that any quantum system which corresponds to the quantization of a classical action which is the integral over time of a lagrangean by Euclidean path integrals would appear to have well defined Green functions. Thus, although the systems which might describe M-Theory in Minkowski space violate the abstract rules of quantum mechanics only by having a bizarre spectral density for the hamiltonian, they are unlikely to have a conventional lagrangean description with respect to an ordinary time variable.

3. Light cone time and black holes

String theory has traditionally been formulated in light cone gauge because, for various technical reasons, one was unable to find another gauge in which the system was obviously a unitary quantum theory. More recently, the light cone gauge has been seen [5] as the framework in which the holographic [6] nature of string theory and M-Theory becomes apparent. Matrix Theory [7] is a nonperturbative formulation of M-Theory in Discrete Light Cone Quantization (DLCQ) on spaces with at least 6 Minkowski dimensions.

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In this section we would like to propose that the success of hamiltonian quantization in light cone gauge is partly due to the absence in light cone gauge of the problems described in the previous section. This leads us to anticipate a problem with any light cone formulation of the theory in 4 noncompact dimensions.

The argument is extremely simple. Light cone energy is defined by

$$\mathcal{E} = \frac{\mathbf{P}^2 + M^2}{P^+} . \tag{3.1}$$

For fixed longitudinal momentum and vanishing transverse momentum, we can therefore write the density of black hole states in light cone energy as

$$\rho(\mathcal{E}) \sim e^{(\frac{P^+\mathcal{E}}{M_P^2})^{\frac{d-2}{2(d-3)}}} . \tag{3.2}$$

Thus, for $d > 4$ the density of states is well behaved, and we can hope to describe the system as some sort of conventional quantum mechanics. In precisely four dimensions the Bekenstein-Hawking formula implies a Hagedorn spectrum for M-Theory in light cone energy.

It is extremely interesting to compare these observations with the problems encountered in compactified DLCQ M-Theory, or Matrix Theory. There it is known that for 7 or more noncompact dimensions the DLCQ description is a quantum field theory, while in 6 dimensions it is [8, 9] a *little string theory*. As described in the next section, the little string theory has a Hagedorn spectrum. Finally, for five dimensions the theory involves gravity, at least in the simplest maximally SUSY case [10]. Note that, qualitatively, the DLCQ density of states seems to mirror that of the uncompactified light cone theory with two fewer dimensions. It is field theoretical down to $d = 7$ and has a Hagedorn form for $d = 6$. There does not, however, appear to be any quantitative mapping of one problem on to the other. The exponents in the energy-entropy relation are completely different. Indeed, it is well known [11] that the DLCQ theory contains many states which must decouple in the large N limit if the limiting theory is to be Lorentz invariant. These undoubtedly are responsible for the dramatically different behavior of the density of states in the DLCQ and uncompactified light cone descriptions. The Lorentz invariant theory contains only the states with energy of order $1/N$ in the large N limit, and the asymptotic density of states in this theory refers to the asymptotics of the *coefficient* $1/N$ in the DLCQ theory.

With five noncompact dimensions, the lightcone hamiltonian of toroidally compactified $DLCQ_N$ M-Theory is that of 11D SUGRA in the presence of an A_N singularity. In this picture the time is that of the rest frame of the singularity. The theory is compactified on a six torus with radii of order the Planck scale. For finite N, the high energy density of states of this theory is dominated by 5D black holes and a conventional Heisenberg quantum mechanics will not exist.

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On the other hand, it has been suggested by Kachru, Lawrence, and Silverstein (KLS) [12] that DLCQ M-Theory compactified on a Calabi-Yau manifold will have fewer and more innocuous extraneous states. These authors also propose that the Matrix description of this theory might be a $3+1$ dimensional field theory. Note that the entropy of such a field theory scales like $E^{3/4}$, *which is precisely the scaling with light cone energy of the entropy of five dimensional black holes.* In the present authors' opinion, this observation supports the suggestion of [12] and should encourage us to search for the relevant $3+1$ dimensional field theory. One clue to its nature might be the fact that the field theoretical nature of the spectrum seems to survive the large N limit. In maximally SUSY Yang Mills theory, there are degrees of freedom with energies as low as $N^{-1/3}$ and a $3+1$ dimensional spectrum, but there do not appear to be any field theoretical degrees of freedom with energies as low as $1/N$. The KLS field theory should have such states.

It is well known that with four noncompact dimensions (i.e. two transverse dimensions) the DLCQ theory ceases to exist. The light cone hamiltonian is the rest frame hamiltonian of toroidally wrapped D7 branes in weakly coupled IIB string theory. This is only a picturesque description, for it is not self-consistent. If we assume an asymptotically locally flat space, the D7 branes have infinite energy (by a BPS formula). However, the gravitational back reaction is infinite and one merely learns that the asymptotically flat ansatz is not self-consistent. Quite likely the theory does not exist at all. The DLCQ theory is really a compactification to 2+1 flat dimensions. Furthermore, like all light cone theories it describes only excited states of the vacuum rather than the vacuum itself. If a 2+1 dimensional theory has four or more supercharges (so that there is an exactly massless scalar in the SUGRA multiplet) then we do not expect there to be many such states. A generic "localized" excitation of the vacuum creates a geometry which is not asymptotically flat. Thus the Hilbert space of the system with asymptotically flat boundary conditions (or even asymptotically locally flat) is very small and contains only a few topological excitations [13].

This problem, which arises in DLCQ, does not appear to have much to do with the apparent absence of a Heisenberg quantum mechanics in the lightcone theory compactified to 4 dimensions. The problem there is an excess of states which do satisfy the asymptotic boundary conditions, rather than a lack of such states. Thus while Matrix Theory may be a useful guide to many properties of M-Theory, we cannot expect to get the physics of low dimensional compactifications right without finding a light-cone description of the Lorentz invariant theory (without compactifying a light-like circle).

3.1 A loose end

A possible objection to the above discussion is that black holes are not stable. Thus they do not really correspond to the eigenspectrum of the hamiltonian. However, the lifetime of black holes goes to infinity as a power of their mass. They are thus extremely narrow resonances and correspond to an enhancement of the density of scattering states. In principle we could have formulated our description of the behavior of Green's functions in terms of complete sets of scattering states and the properties of the S-matrix.

4. Hamiltonian description of little string theories

In section 2 we briefly discussed the convergence of the formal expression (2.2) in the case of local field theories, and noted that there appeared to be no problem with it in this case. However, it was realized in the past few years that there are also non-local field theories which can be decoupled from gravity, in particular little string theories. Although decoupled from gravity, these behave in many ways like critical string theories. In this section we will analyze the behavior of these theories at high energies, and we will argue that they have a Hagedorn density of states. Therefore, using the arguments of section 2, it is not clear how to define local operators in these theories using the usual time variable, but a light-cone description of these theories in terms of an ordinary lagrangean quantum mechanics does seem to make sense. We will discuss in detail only the case of little string theories with 16 supercharges in 6 dimensions of type A_{k-1}, but we expect the conclusions to be more general.

The construction of little string theories with 16 supercharges in 6 dimensions was discussed in [9]. The original definition of these theories involved looking at k NS 5-branes in type IIA (for the $\mathcal{N} = (2,0)$ little string theories) or type IIB (for the $\mathcal{N} = (1,1)$ little string theories) string theories, and taking the limit of $g_s \to 0$ with the string scale α' constant. While this construction provides evidence for the existence of such little string theories, it does not allow for explicit computations in these theories. Two independent methods for making direct computations in these theories were developed in the past two years, and we will use both of them to compute the density of states at high energies.

4.1 The equation of state from the holographic description

The first method we will use is the holographic description of the little string theories [14], which is a generalization of the AdS/CFT correspondence [2, 15, 16]. The little string theories are claimed to be dual to a background of M-Theory or string theory which, at a large radial coordinate, asymptotes to a linear dilaton background of string theory (with a string metric which is the standard metric on $\mathbb{R}^7 \times S^3$). It is easy to generalize this description also to the case of finite energy density or temperature. As in the case of the AdS/CFT correspondence [17], the relevant background (at least at high energy densities) is the near-horizon background of near-extremal NS5-branes. The simplest way to derive this background is just to take the $g_s \to 0$ limit in the background of a near-extremal 5-brane [18]. The result is very similar to the background described in [14], but with the linear dilaton direction and the time direction replaced by an SL(2)/U(1) black hole (with an appropriate SL(2) level so that the total central charge is $\hat{c} = 10$). The black hole background also has a varying dilaton with the string coupling going to zero far from the horizon. If we start with an energy density $E/V = \mu M_s^6$ on the 5-brane, the string coupling at the horizon is $g_s^2 \sim k/\mu$. Since for large k the curvature of this background is small, it follows that for $\mu \gg k \gg 1$ we can trust the supergravity description of this background [18], and it provides a holographic dual for the little string theories with this energy density.[4]

In particular, we can use this description to compute the equation of state of the little string theories from the Bekenstein-Hawking formula (as was done for the case of the AdS/CFT correspondence in [17]), and we find that the equation of state is [18]

$$E = \frac{M_s}{\sqrt{6k}} S, \tag{4.1}$$

where E and S can be taken to be either the total energy and entropy or the energy and entropy densities (since the formula does not depend on the volume). As was noted in a similar context in [20], this is the same density of states as that of a free string theory with a string scale of $k\alpha'$ and $c = 6$, or of a free string theory with a string scale of α' and $c = 6k$, even though the theory does not seem to be a free string theory; an interpretation as a theory with a string scale α' seems more likely since the theory has a T-duality symmetry corresponding to this scale [9]. In any case, for our purposes it is enough to note that at high energy densities we get the equation of state (4.1), which is a Hagedorn density of states with a Hagedorn temperature of $T_H = M_s/\sqrt{6k}$ (signifying that the canonical ensemble can only be defined below this temperature).

The fact that (at least for large volumes compared to the string scale) the equation of state does not depend on the volume suggests that, as in M-Theory, the high energy density of states is dominated by the states of a single object. In little string theory we believe that the analog of the black hole is a single long string. This seems to be the message of the Hagedorn spectrum. Note that this is not strictly true in free string theory, since there the numbers of strings in each string state are an infinite set of conserved quantities. However, when interactions are turned on, multiple string states can convert into a single long string and this has more entropy. In the interacting little string theory we should expect this phenomenon to occur as well. We will provide some supporting evidence for this from the DLCQ description below, by computing the Hagedorn description via an independent argument which shows that it can naturally arise just from single string states. As noted above, the corresponding phenomenon for black holes in asymptotically Minkowski spacetime

[4] For smaller energy densities we expect this background to be corrected, at least in the $\mathcal{N} = (2, 0)$ case where at low temperatures and energy densities the solution should become localized in the eleventh direction [19].

is the Gross Perry Yaffe [1] instability of the translation invariant thermal ensemble to the formation of a single large black hole. In the little string theories we do not expect a similar localization of the generic high energy states, but they still seem to correspond to single objects, unlike the local field theory case.

We argued for the full M-Theory that the existence of black holes precluded the existence of local operators, which couple only to a small subset of the high energy states. We believe that the same is true in the little string theory, as a consequence of the fact that the generic high energy state is a single big string. In field theory (on a large but finite torus) the generic state of high energy is a translation-invariant gas. But in an interacting string gas in any finite volume, once the energy is taken large enough, the density of strings is such that overlaps are inevitable, and in the presence of interactions the high entropy single string state will be preferred. In a gas, it is easy to construct operators which create only one of the constituents from the vacuum. On the other hand, if perturbative string theory is any guide, it is very difficult to construct operators with matrix elements between the vacuum and only a few of the highly degenerate excited string states.

Our arguments here have necessarily been quite heuristic because we do not have a good description of the eigenstates of little string theories. Nonetheless, combined with the supporting evidence from the DLCQ and holographic pictures, and especially the calculation of Peet and Polchinski [21] which suggests that correlation functions of little string theory operators in momentum space do not have Fourier transforms, our description of the physics of this system seems plausible.

4.2 The equation of state from the DLCQ description

A completely different description of the little string theories is their discrete light-cone quantization, which was described in [22, 23] for the case with $\mathcal{N} = (2,0)$ supersymmetry and in [24] for the case with $\mathcal{N} = (1,1)$. In both cases the description of the theory with light-like momentum $P^+ = N/R$ is given by a $1+1$ dimensional conformal theory with $c = 6Nk$, compactified on a circle of radius $\Sigma = 1/RM_s^2$. Conformal invariance dictates the equation of state of these theories at high energies (above the scale $1/\Sigma$) to be

$$E_{DLCQ} = 6Nk\Sigma T^2 \,, \qquad S = 6Nk\Sigma T \,, \tag{4.2}$$

or

$$E_{DLCQ} = S^2/6Nk\Sigma \,. \tag{4.3}$$

As in the previous section, we can easily translate this into an equation of state for the full space-time theory (a similar procedure was carried out for Matrix black holes in [25]). In the absence of transverse momentum we have $E_{DLCQ}P^+ = E^2$, so we get

$$E = \sqrt{E_{DLCQ}N/R} = \frac{M_s}{\sqrt{6k}}S \,, \tag{4.4}$$

which is exactly the same relation as (4.1). Note that all factors of N and R dropped out of this expression, as well as any dependence on the volume of space, without the necessity of taking the large N limit; this happens here because of special properties of the DLCQ of the little string theories and would not be true in general.

The computation above gives the high-energy density of states in the DLCQ theory; unfortunately this is not what we are interested in for the Lorentz-invariant theory, for which only states whose energies are of order $1/N$ are relevant (obviously for large N these energies will become smaller than the scale $1/\Sigma$ above which our previous computation was valid). Luckily, as in the case of the DLCQ description of type IIA string theory [26, 27, 28], we can argue that these theories have states whose energy scales like $1/N$ which obey the same equation of state as the full theory. In the case of type IIA string theory, these states involved "long string" states which changed by a $\mathrm{U}(N)$ gauge transformation when going around the compact circle. For free type IIA string theory the DLCQ involved a free CFT of central charge $\hat{c} = 8N$, while the "long string states" (for the lowest-lying states for which the gauge transformation was equivalent to a permutation of order N of the eigenvalues of the $\mathrm{U}(N)$ adjoint matrices) involved a CFT of central charge $\hat{c} = 8$ but compactified on a circle of radius $\tilde{\Sigma} = N\Sigma$; since the formula (4.3) depends only on the product $c\Sigma$ these states obey the same equation of state as that of the full theory. We would like to suggest that a similar mechanism holds also in the DLCQ description of the little string theories. For the $\mathcal{N} = (2,0)$ case this involves the Higgs branch of a $\mathrm{U}(N)$ gauge theory while for the $\mathcal{N} = (1,1)$ case it involves the Coulomb branch of a $\mathrm{U}(N)^k$ gauge theory. At least in the latter case it is clear that the theory includes "long string" states with energies of order $1/N$ just like in the full type IIA string theory, and it seems likely that the central charge for the theory describing the "long strings" will be $1/N$ of the total central charge. A complete analysis of these "long string" states will be presented elsewhere. Unfortunately, since these states are strongly interacting (unlike the case discussed above of weakly coupled type IIA string theory), it is not clear if we can really trust this description for computing the density of states. In particular, it is not obvious that the "long string" states are adequately described by a local CFT. However, it seems plausible that "long string" states do exist and obey an equation of state similar to (4.3) (up to possible numerical factors). We view this as additional evidence for the validity of (4.1) in the little string theories, and for the entropy being dominated by single-string states.

4.3 Discussion

Let us now discuss the consequences of (4.1) for the description of the little string theories. As discussed in section 2, this behavior implies that correlation functions of standard Heisenberg operators do not exist in these theories, at least when the time difference between the operators is smaller than the Hagedorn scale $1/T_H$. Indeed, using the holographic description [14] of the little string theories, Peet and Polchin-

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ski [21] have provided independent evidence that the correlation functions of these theories are not Fourier transformable and do not obey the rules of quantum field theory. This supplements the arguments based on T-duality.

Thus, we expect the DLCQ description of these theories (or perhaps a direct large N limiting version of it) to be the only lagrangean quantum theory which computes the correlation functions and eigenspectrum of the theory.

We want to emphasize the way in which the DLCQ analysis agrees with the Bekenstein-Hawking analysis of these theories. DLCQ predicts a Hagedorn spectrum in ordinary Lorentz frames in a very robust way. The argument depends only on general properties of $1+1$ dimensional field theories. The only possible loophole in the argument is that the spectrum of states whose energies in the large N limit are of order $1/N$ might not be field theoretic. We believe we have provided plausible arguments which close this loophole, though more work is necessary to elucidate the nature of long string states in these interacting theories. The success of the Bekenstein-Hawking argument in predicting the correct density of states in these systems motivated us to apply it to quantum gravity in the bulk of this paper.

5. Conclusions and questions

What are we to make of the failure of Heisenberg quantum mechanics in light cone gauge for gravitational theories in four dimensions? Does this spell the end of the search for a nonperturbative lagrangean formulation of M-Theory? There are several possibilities:

1. Our reaction to the nonexistence of Green's functions has been too violent, at least in the case of a Hagedorn spectrum. For example, in first quantized string theory, the system has a Hagedorn spectrum in ordinary Lorentz frames. Nonetheless a covariant lagrangean and hamiltonian formalism exists in conformal gauge. The time variable in this formalism is not connected to *any* spacetime variable. There are two reasons to be suspicious of the possibility of generalizing this formalism to a nonperturbative interacting theory. First of all, the light cone gauge formulation of the theory does not have any problems, and the covariant formalism bears a very close resemblance to it. Secondly, the free string theory is completely integrable and there are natural operators which communicate only with single states among the Hagedorn spectrum. One should however emphasize as well that the divergence of Green's functions is much less dramatic for a Hagedorn spectrum than it is for black holes far from extremality. In particular, in the Hagedorn case, Euclidean Green's functions exist as long as all time intervals are sufficiently long. Furthermore, the little string theories give us an example of interacting, non-gravitational systems with a Hagedorn spectrum.

2. Perhaps, as in Matrix Theory, light cone M-Theory in four dimensions can be formulated as some sort of compactification of a theory with an auxiliary Lorentz invariance under which the light cone time variable of M-Theory transforms as the time component of a Lorentz vector. Then we can formulate the theory in the light cone frame of the auxiliary Lorentz group and deal with the Hagedorn spectrum by the same trick which works in first quantized string theory. This is the proposal of [22, 23] for treating little string theories. It is not clear how such a proposal could work for four dimensional M-Theory without compactifying a light-like direction (which we cannot do in this case as discussed in section 3). In a DLCQ theory, the auxiliary Lorentz group relates the lightcone hamiltonian to the charges of longitudinally wrapped branes. These extra unwanted "momentum" quantum numbers disappear into the ultraviolet in the large N limit, so the auxiliary Lorentz group does not act on the states which survive in the large N limit (the "momentum" states all have an infinite energy in the limit of a non-compact light-cone description). Thus, this symmetry should not exist in the exact light-cone description of the four dimensional theory. Nonetheless, we should emphasize again that the closest analog to a putative four dimensional light cone M-Theory is the little string theory, and this *does* have a DLCQ description which is an ordinary quantum mechanical theory.

3. Perhaps M-Theory with four Minkowski dimensions can only be defined as the limit of M-Theory with AdS asymptotics. We have pointed out above that the current understanding of the AdS/CFT correspondence does not furnish us with a prescription for extracting the Minkowski S-Matrix from the AdS theory, but perhaps this difficulty can be overcome. The most serious objection we can find to such a proposal is that the most likely candidate theory would be of the form $AdS_2 \times S^2 \times X$, but AdS_2 theories seem to be topological [29].

4. The real world is not Minkowski space but rather a cosmological space time. Perhaps we should be searching for the fundamental formulation of M-Theory only in the context of closed cosmologies (cosmologies where all space-like slices are compact). In this case we do not expect the notion of hamiltonian to have a fundamental significance. Time evolution is a concept which is recovered only in a semiclassical approximation. The problems we seem to have with formulating a Heisenberg picture quantum mechanics may signal a breakdown of this semiclassical approximation rather than a fundamental problem. We have to admit that we don't understand how this could be the case.

Whatever the resolution of these difficulties, we cannot end this paper without making note of two significant points. The first is the privileged position of four dimensions in this discussion. Gravitational theories with fewer than four (Minkowski) dimensions do not have many states. This has been advanced in [13] as a reason why they are not the endpoint of cosmological evolution. On the other hand, the Bekenstein-Hawking formula tells us that in some sense four dimensional Minkowski spacetimes have more states at a given asymptotic energy level than higher dimensional spaces (and all Minkowski spaces have more states than any AdS space). Perhaps this observation will be the key to understanding why the world we observe is four dimensional.

Our final comment is to emphasize the similarity between the high energy spectra of four dimensional light cone M-Theory and of little string theories in an ordinary reference frame. This suggests that, although the light cone quantum mechanics describing four dimensional M-Theory is not a conventional lagrangean theory, it may be some sort of little string theory. This fascinating conjecture is an obvious direction for future work.

Acknowledgments

We are grateful to M. Berkooz, W. Fischler, D. Kutasov, N. Seiberg and Sasha Zamolodchikov for valuable discussions. This work was supported in part by the DOE under grant number DE-FG02-96ER40559.

References

[1] D.J. Gross, M.J. Perry and L.G. Yaffe, *Instability of Flat Space at Finite Temperature, Phys. Rev.* **D 25** (1982) 330.

[2] J.M. Maldacena, *The Large N Limit of Superconformal Field Theories and Supergravity, Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[3] S.S. Gubser, I.R. Klebanov and A.W. Peet, *Entropy and Temperature of Black 3-branes, Phys. Rev.* **D 54** (1996) 3915 [hep-th/9602135];
I.R. Klebanov and A.A. Tseytlin, *Entropy of Near Extremal Black p-branes, Nucl. Phys.* **B 475** (1996) 164 [hep-th/9604089].

[4] T. Banks, M. Douglas, G. Horowitz and E. Martinec, *AdS Dynamics from Conformal Field Theory,* hep-th/9808016.

[5] C. Thorn, *Reformulating String Theory with the* $1/N$ *Expansion,* published in *Sakharov Conf. on Physics*, Moscow (1991) 447, hep-th/9405069.

[6] G. 't Hooft, *Dimensional Reduction in Quantum Gravity,* published in Salamfest 1993, 284, gr-qc/9310026;
L. Susskind, *The World as a Hologram, J. Math. Phys.* **36** (1995) 6377 [hep-th/9409089].

[7] T. Banks, W. Fischler, S. Shenker and L. Susskind, *M-Theory as a Matrix Model: A Conjecture, Phys. Rev.* **D 55** (1997) 5112 [hep-th/9610043].

[8] M. Berkooz, M. Rozali and N. Seiberg, *Matrix Description of M-Theory on T^4 and T^5, Phys. Lett.* **B 408** (1997) 105 [hep-th/9704089].

[9] N. Seiberg, *New Theories in Six Dimensions and Matrix Description of M-Theory on T^5 and $T^5/\mathbb{Z}_2$, Phys. Lett.* **B 408** (1997) 98 [hep-th/9705221].

[10] N. Seiberg, *Why is the Matrix Model Correct ? Phys. Rev. Lett.* **79** (1997) 3577 [hep-th/9710009].

[11] T. Banks, *Matrix Theory, Nucl. Phys.* **67** *(Proc. Suppl.)* (1998) 180 [hep-th/9710231].

[12] S. Kachru, A. Lawrence and E. Silverstein, *On the Matrix Description of Calabi-Yau Compactifications, Phys. Rev. Lett.* **80** (1998) 2996 [hep-th/9712223].

[13] T. Banks and L. Susskind, *The Number of States of Two Dimensional Critical String Theory, Phys. Rev.* **D 54** (1996) 1677 [hep-th/9511193].

[14] O. Aharony, M. Berkooz, D. Kutasov and N. Seiberg, *Linear Dilatons, NS5-branes and Holography, J. High Energy Phys.* **9810** (1998) 004 [hep-th/9808149].

[15] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge Correlators from Noncritical String Theory, Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[16] E. Witten, *Anti-de Sitter Space and Holography, Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[17] E. Witten, *Anti-de Sitter Space, Thermal Phase Transition, and Confinement in Gauge Theories, Adv. Theor. Math. Phys.* **2** (1998) 505 [hep-th/9803131].

[18] J.M. Maldacena and A. Strominger, *Semiclassical Decay of Near Extremal Five-branes, J. High Energy Phys.* **9712** (1997) 008 [hep-th/9710014].

[19] N. Itzhaki, J.M. Maldacena, J. Sonnenschein and S. Yankielowicz, *Supergravity and the Large N Limit of Theories with Sixteen Supercharges, Phys. Rev.* **D 58** (1998) 46 [hep-th/9802042].

[20] J.M. Maldacena, *Statistical Entropy of Near Extremal Five-branes, Nucl. Phys.* **B 477** (1996) 168 [hep-th/9605016].

[21] A.W. Peet and J. Polchinski, *UV/IR Relations in AdS Dynamics, Phys. Rev.* **D 59** (1999) 065006 hep-th/9809022.

[22] O. Aharony, M. Berkooz, S. Kachru, N. Seiberg and E. Silverstein, *Matrix Description of Interacting Theories in Six Dimensions, Adv. Theor. Math. Phys.* **1** (1998) 148 [hep-th/9707079].

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[23] E. Witten, *On the Conformal Field Theory of the Higgs Branch, J. High Energy Phys.* **9707** (1997) 003 [hep-th/9707093].

[24] O.J. Ganor and S. Sethi, *New Perspectives on Yang-Mills Theories with Sixteen Supersymmetries, J. High Energy Phys.* **9801** (1998) 007 [hep-th/9712071].

[25] T. Banks, W. Fischler, I.R. Klebanov and L. Susskind, *Schwarzschild Black Holes from Matrix Theory, Phys. Rev. Lett.* **80** (1998) 226 [hep-th/9709091]; *J. High Energy Phys.* **9801** (1998) 008 [hep-th/9711005].

[26] L. Motl, *Proposals on Nonperturbative Superstring Interactions,* hep-th/9701025.

[27] T. Banks and N. Seiberg, *Strings from Matrices, Nucl. Phys.* **B 497** (1997) 41 [hep-th/9702187].

[28] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Matrix String Theory, Nucl. Phys.* **B 500** (1997) 43 [hep-th/9703030].

[29] A. Strominger, *AdS_2 Quantum Gravity and String Theory, J. High Energy Phys.* **01** (1999) 007 [hep-th/9809027];
J. Maldacena, J. Michelson and A. Strominger, *Anti-de Sitter Fragmentation, J. High Energy Phys.* **02** (1999) 011 [hep-th/9812073].